Approximation Algorithms

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Outline of the tutorial

Part 1:

- Approximation algorithms: an example and definitions
- Clustering problems: k-center and k-median
- Bottleneck problems: 2-approximation for k-center
- Local search: $(3 + \epsilon)$-approximation for k-median

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- Approximation algorithms: an example and definitions
- Clustering problems: k-center and k-median
- Bottleneck problems: 2-approximation for k-center
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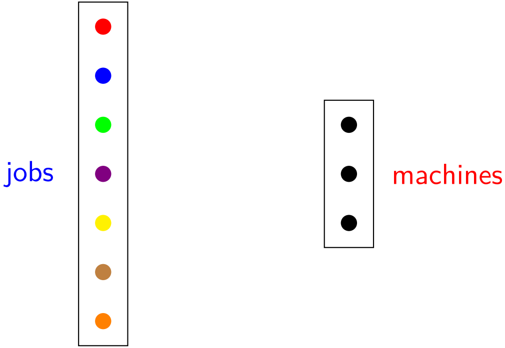
Part 2:

- Probabilistic strategies: 0.5-approximation for MaxSAT
- Linear programming: 0.63-approximation for the MaxSAT
- Mixed strategies: 0.75-approximation for the MaxSAT
- Closing remarks

Scheduling in identical machines

Given: m machines
n jobs
processing time t_i of job i $(i = 1, \ldots, n)$



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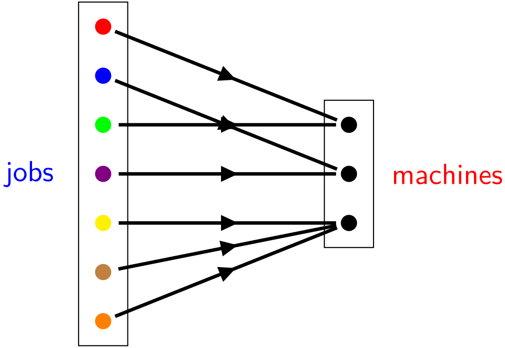

a scheduling is a partition $\{M_1, \ldots, M_m\}$ of $\{1, \ldots, n\}$.

Example 1

$m = 3$ and $n = 7$

t_1	t_2	t_3	t_4	t_5	t_6	t_7
3	2	7	5	1	6	2

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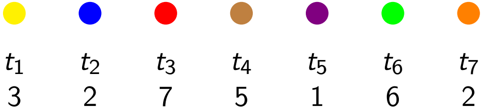

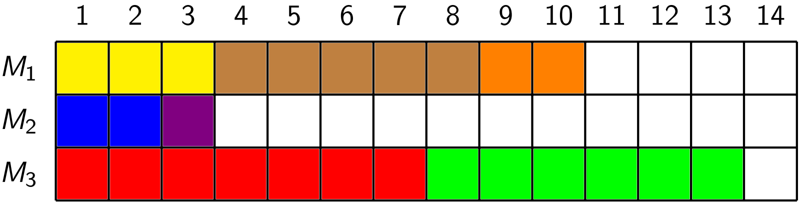

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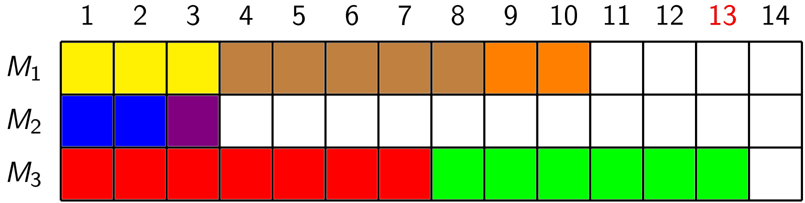

partition $\{\{1,4,7\},\{2,5\},\{3,6\}\} \Rightarrow$ makespan $= 13$

Example 2

$m = 3$ and $n = 7$

t_1	t_2	t_3	t_4	t_5	t_6	t_7
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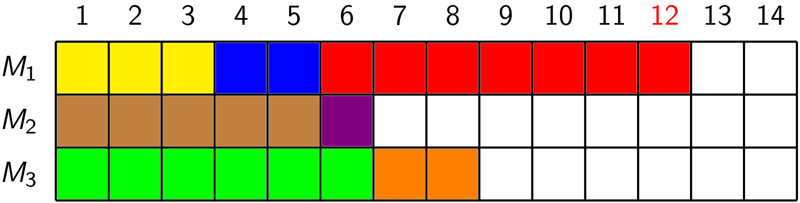

partition $\{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}\} \Rightarrow$ makespan $= 12$

Problem

Find a scheduling with minimum makespan.

t_1	t_2	t_3	t_4	t_5	t_6	t_7
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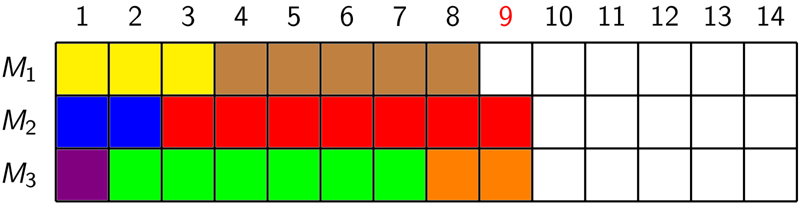
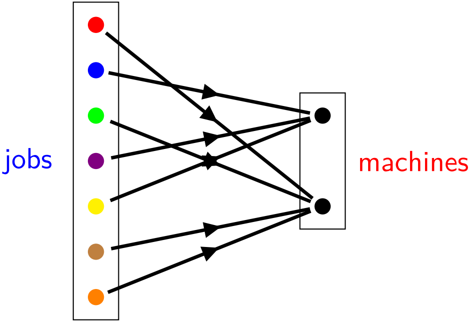


partition $\{\{1,4\},\{2,3\},\{5,6,7\}\} \Rightarrow$ makespan $= 9$

Hardness

Scheduling on two machines: given n and t,
find a scheduling for two machines with minimum makespan.



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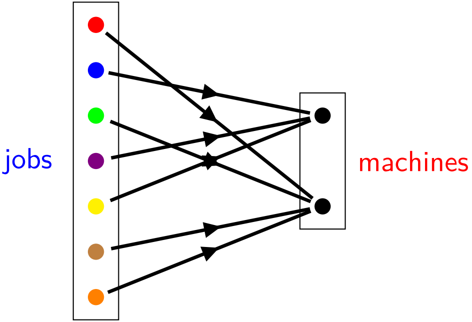

Partition: Given a set S numbers,
decide if there is a subset $X \subseteq S$ such that $\sum_{s \in X} s = \sum_{s \in S \setminus X} s$.

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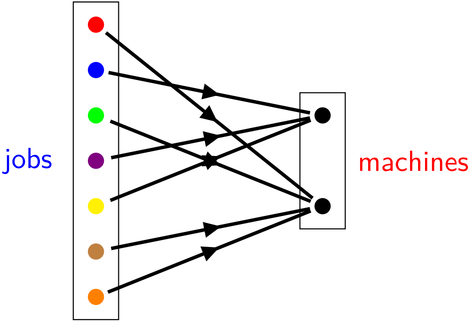

Even this particular case is NP-hard, that is,
if there is a polynomial-time algorithm for this case, then $\mathrm{P} = \mathrm{NP}$.

Graham's algorithm

Assign each job, one by one, to the first available machine.

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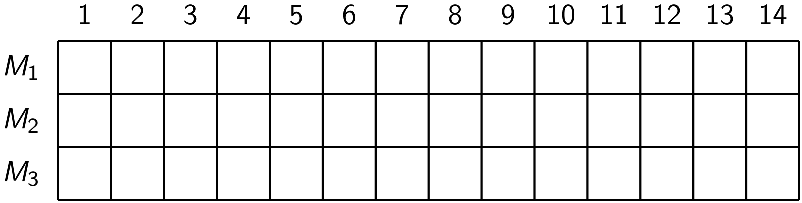

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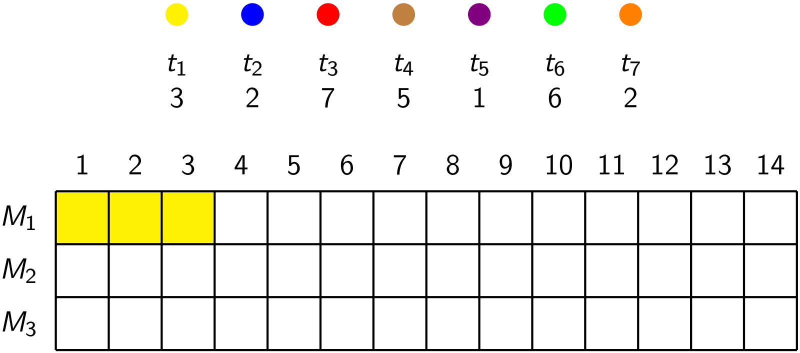

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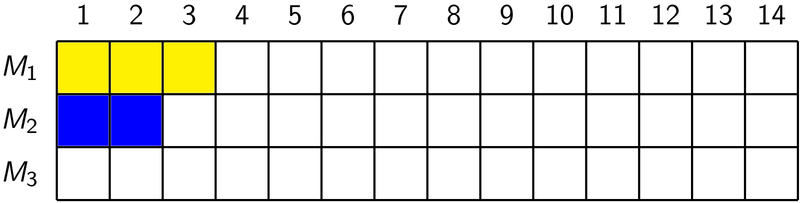

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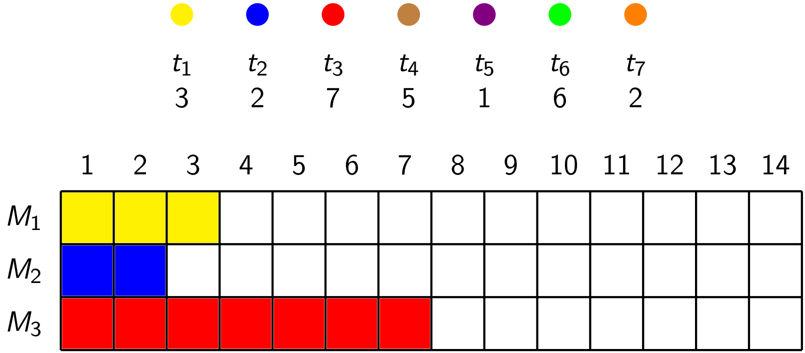

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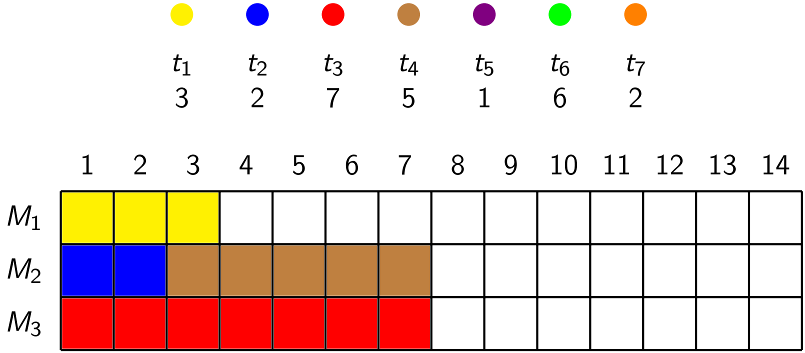

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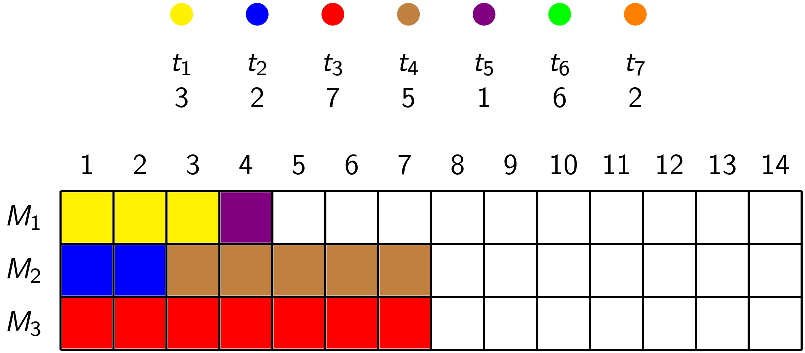

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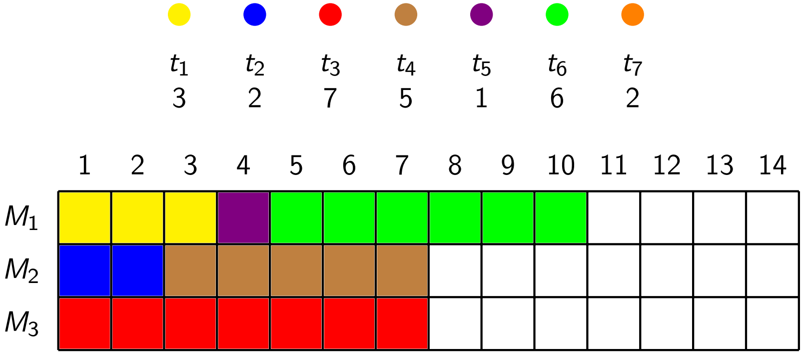

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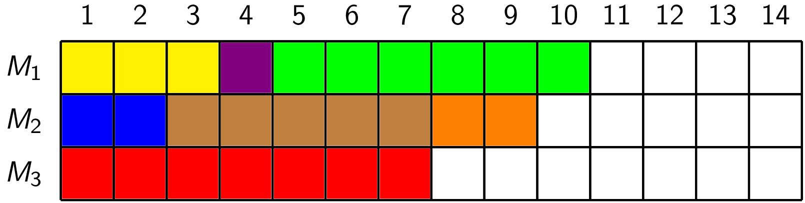

Graham's algorithm is polynomial.

How bad can the makespan be?

Bounds on OPT

OPT = minimum makespan

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$$\mathrm{OPT} \geq \max\{t_1, t_2, \ldots, t_n\}$$

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- Balanced distribution:

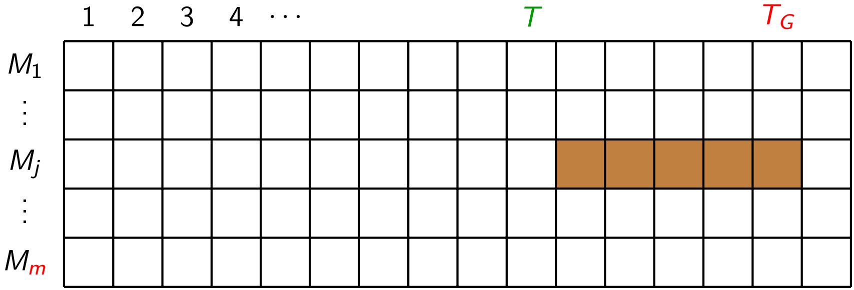
$$\mathrm{OPT} \geq \frac{t_1 + t_2 + \cdots + t_n}{m}$$

Makespan of Graham's scheduling

T_G: makespan of the algorithm

job i: job that finishes at time T_G

time T: time previous to the starting time of job i



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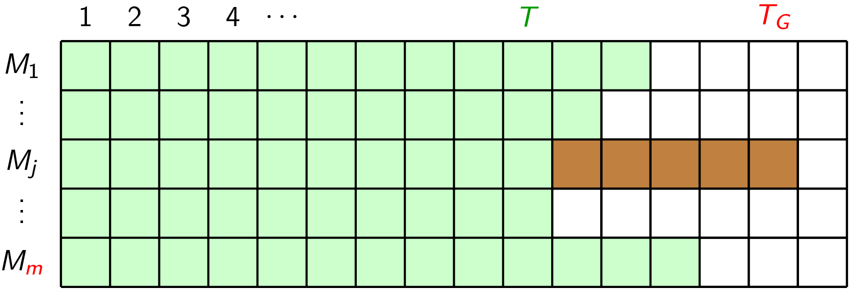
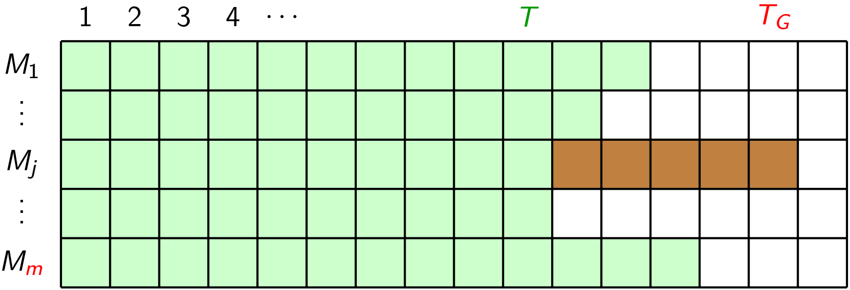


$$T \cdot m < t_1 + \cdots + t_n \quad \Rightarrow \quad T < \frac{t_1 + \cdots + t_n}{m} \leq \mathrm{OPT}$$

Quality of Graham's scheduling



$$T_G = T + t_i$$
$$< \mathrm{OPT} + \max\{t_1, \ldots, t_n\}$$
$$\leq \mathrm{OPT} + \mathrm{OPT}$$
$$= 2\,\mathrm{OPT}$$

Approximation algorithm

Context:

- Π: optimization problem (minimization)
- $\mathrm{cost}(S, I)$: cost of the feasible solution S for instance I of Π
- $\mathrm{OPT}(I)$: minimum cost of a feasible solution for instance I of Π

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given any instance I for Π, produces a feasible solution for I.

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Approximation algorithm

if A is polynomial and there exists a number $\alpha \geq 1$ such that

$$\mathrm{cost}(A(I), I) \leq \alpha \, \mathrm{OPT}(I) \text{ for every instance } I \text{ of } \Pi,$$

then A is an α-**approximation**.

Graham's algorithm

Input: positive integers m and n, and an array $t[1..n]$
Output: a scheduling of $\{1, \ldots, n\}$ in m machines.

Algorithm Graham (m, n, t)

1. for $j := 1$ to m do
2. $\quad M_j := \emptyset$
3. $\quad T_j := 0$ $\qquad \triangleright$ available instant for machine j
4. for $i := 1$ to n do
5. $\quad$ let k be such that T_k is minimum
6. $\quad M_k := M_k \cup \{i\} \qquad T_k := T_k + t_i$
7. return $\{M_1, \ldots, M_m\}$

Graham's algorithm is a 2-approximation.

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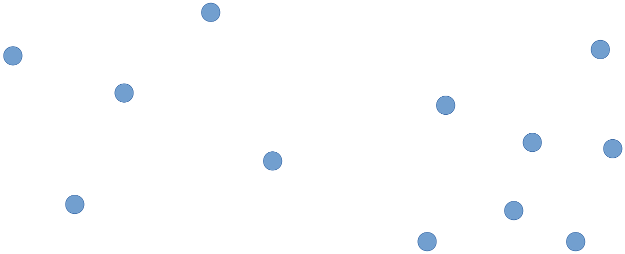
Exercise 1:
What if we schedule the jobs in decreasing order of the processing time?

Clustering problems

Classical k-center

Given:

- a positive integer k,
- a set V of elements, and
- a function $d : V \times V \to \mathbb{Q}^+$,



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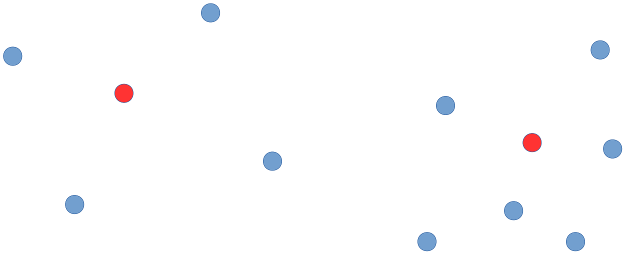
find a set $S \subseteq V$ with $|S| = k$ that ...



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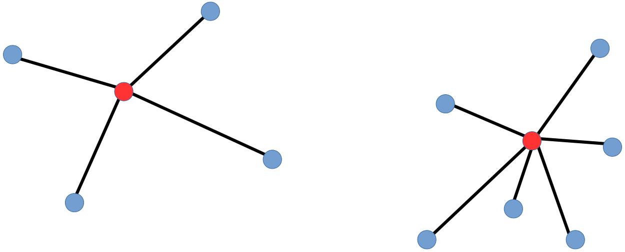

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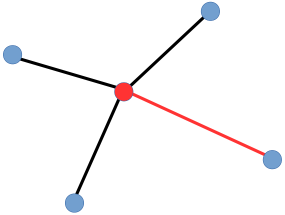
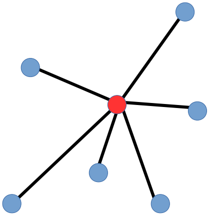
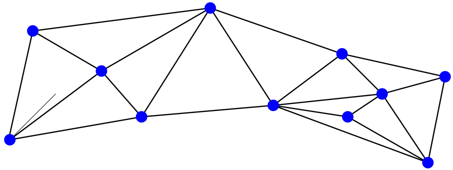




Hardness

Graph G



Dominating set: set S of vertices of G such that each vertex of G is in S or has a neighbor in S.

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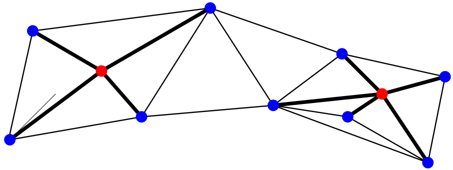

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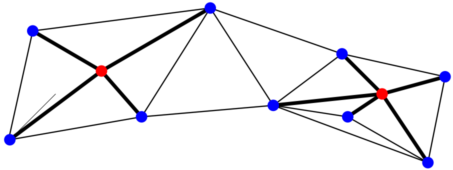

Dominating set: set S of vertices of G such that each vertex of G is in S or has a neighbor in S.

Reduction to a k-center instance: take the same k, $V = V(G)$ and $d(x, y) = 1$ if x and y are adjacent in G, and $d(x, y) = M$ otherwise.

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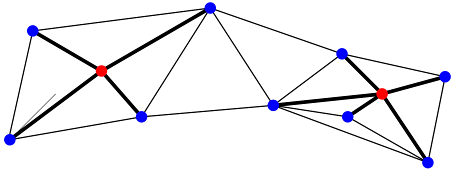
Graph G



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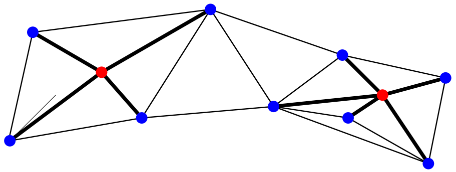
Theorem

There is a dominating set of size k in G if and only if there is a k-center solution of radius 1 for the instance (k, V, d).

Inapproximability

Graph G



The **k-center instance** is $V = V(G)$ and
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Hard even to approximate:

An α-approximation for k-center with $\alpha < M$ solves dominating set.

Inapproximability

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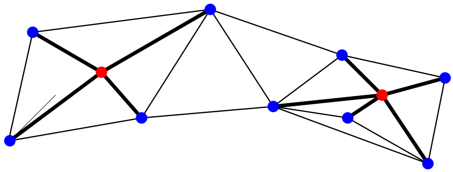

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An α-approximation for k-center with $\alpha < M$ solves dominating set.

Theorem

There is no α-approximation for the k-center problem, unless $\mathrm{P} = \mathrm{NP}$.

Too hard to approximate?

What to do?

Restrict attention to specific classes of instances.

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A function $d : V \times V \to \mathbb{Q}^+$ is a metric if, for every $x, y, w \in V$,

- $d(x, y) = d(y, x)$ (symmetry)
- $d(x, y) \leq d(x, w) + d(w, y)$ (triangle inequality)

Such a function d is called a distance function.

Metric instances

If d is a distance function, then the instance is metric.

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The **k-center instance built from the dominating set instance:**
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Exercise 2:

How does the previous inapproximability result apply to the metric k-center?

Bottleneck problems

- Optimal value is the length of an edge

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Example of bottleneck problem: k-center

Instance: positive integer k, set V, and distance function d on V.

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Example of bottleneck problem: k-center

Instance: positive integer k, set V, and distance function d on V.

G: complete graph on V with length $\ell(uv) = d(u, v)$ for each edge uv.

Bottleneck problems

Metric k-center: positive integer k and
complete graph G on V with length $\ell(uv) = d(u, v)$ for each edge uv.

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H^2: the square of H (add edges between vertices at distance 2 in H)

Bottleneck problems

Metric k-center: positive integer k and complete graph G on V with length $\ell(uv) = d(u,v)$ for each edge uv.

Idea for an algorithm:

1. Sort edges of G by length: $\ell(e_1) \leq \cdots \leq \ell(e_m)$.
2. Threshold graph G_i : $G[E_i]$ where $E_i := \{e_1, \ldots, e_i\}$.
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A maximal independent set in G_i^2 is a set of centers in G of radius $2\,\ell(e_i)$.

Bottleneck problems: metric k-center

Algorithm GHS (k, G, ℓ) $\triangleright$ Gonzalez '85, Hochbaum and Shmoys '85

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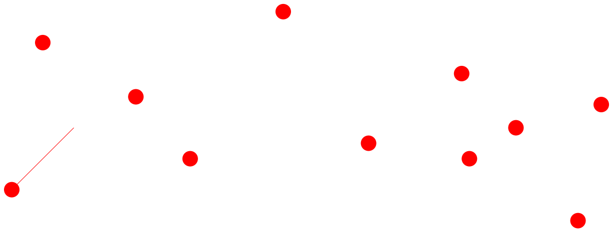

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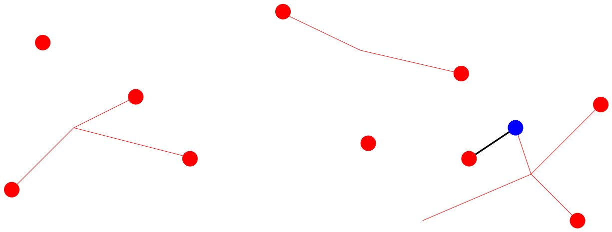

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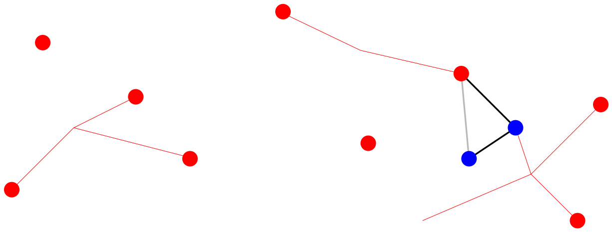

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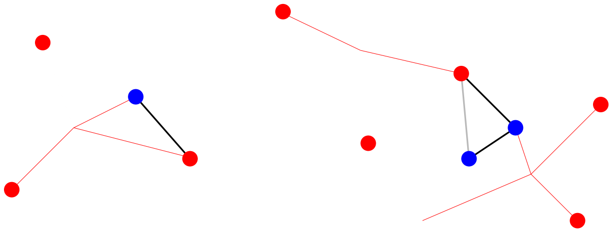

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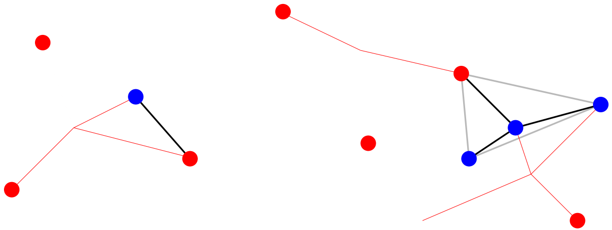

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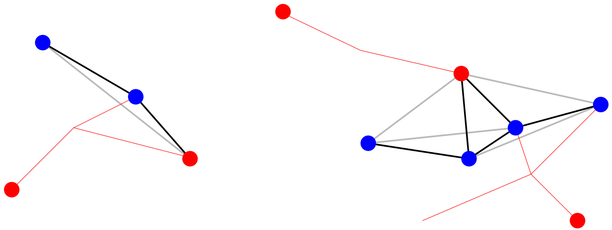

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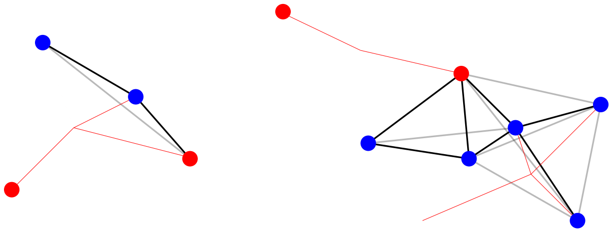

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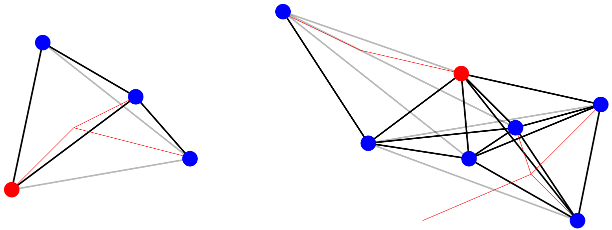

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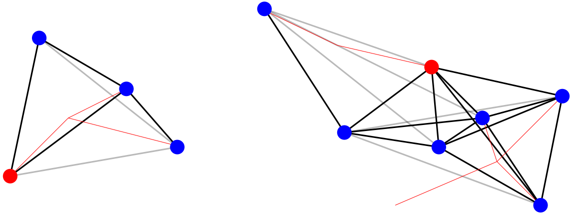

The radius of M_i is at most $2\,\ell(e_i)$.

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Hence the radius of M_i is at most $2\,\ell(e_i) \leq 2\,\ell(e_{i^*}) = 2\,\mathrm{OPT}$.

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Exercise 3:

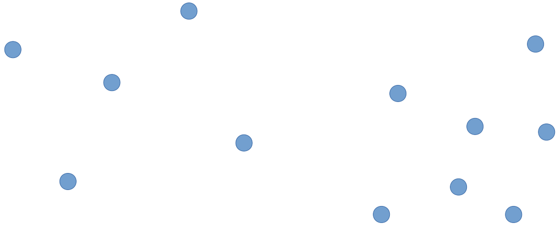
Is there an α-approximation with $\alpha < 2$ for the metric k-center?

Clustering problems

Classical k-median

Given:

- a positive integer k,
- a set V of elements, and
- a function $d : V \times V \to \mathbb{Q}^+$,



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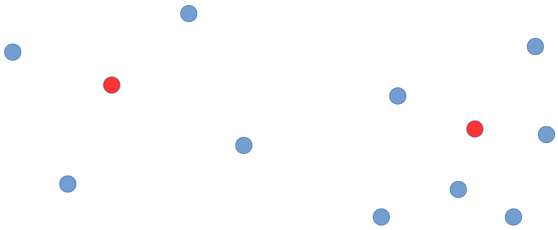

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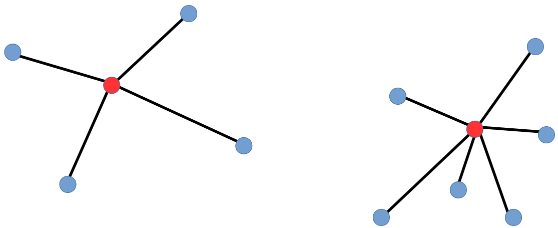

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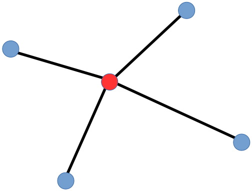

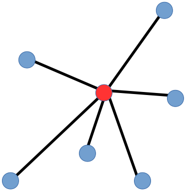

There is no α-approximation
for constant $\alpha > 1$ unless $\mathrm{P} = \mathrm{NP}$.

Local search: metric k-median

k-median instance: positive integer k, set V, and distance function d.

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Let S be a subset of V of size k. Let $u \in S$ and $v \notin S$.

Pair (u, v) is an improving swap for S if $S' = S - u + v$ has better cost:

$$\sum_{u \in V} \min_{v \in S} d(u, v) > \sum_{u \in V} \min_{v \in S'} d(u, v).$$

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Algorithm AGKMMP (k, V, d) ▷ Arya et al. '01

1. let S be an arbitrary set of k elements of V
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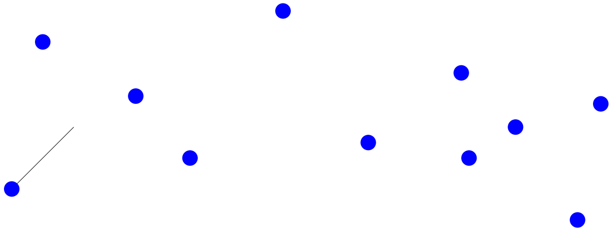

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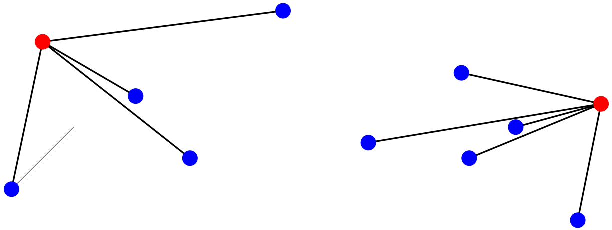

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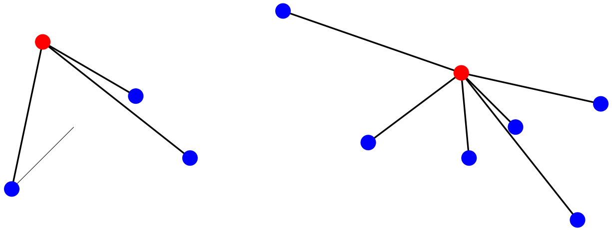

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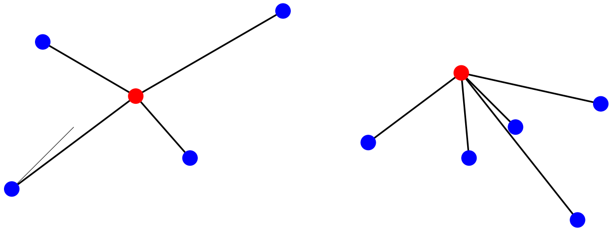

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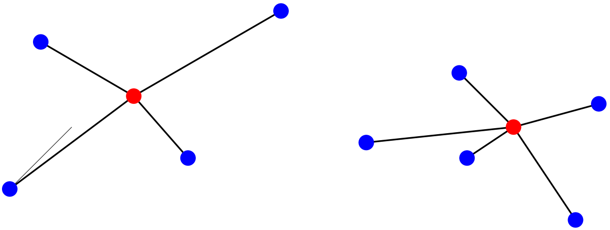

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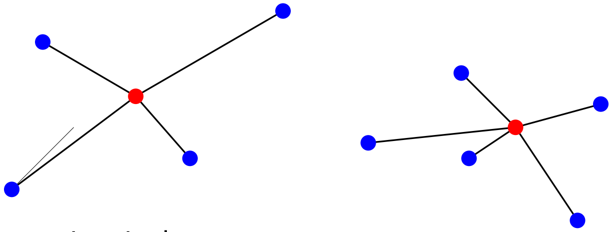
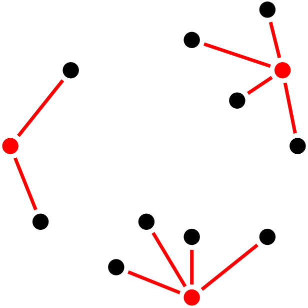


This is a 5-approximation!

Sketch of the approximation analysis

Let S^* be an optimal solution and $N^*(o)$ be the clients of o in S^*



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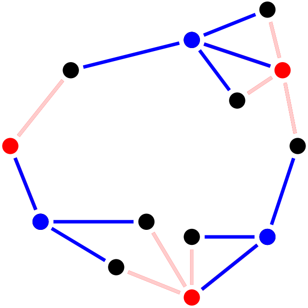

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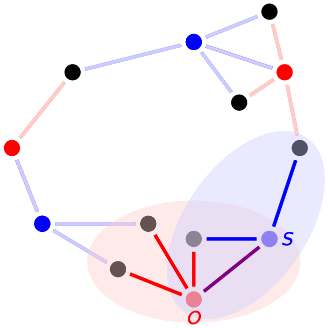

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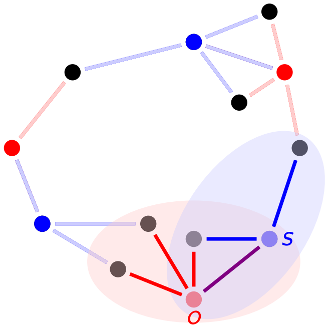

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Each $o \in S^*$ is captured by at most one element from S.

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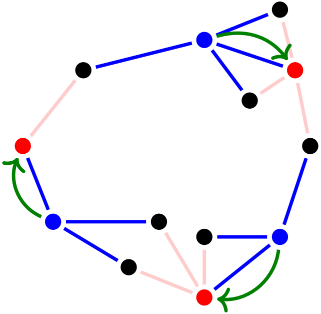
Assume that each $s \in S$ captures exactly one element o from S^*.



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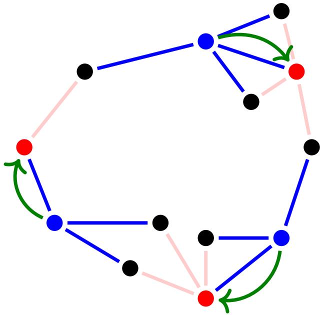

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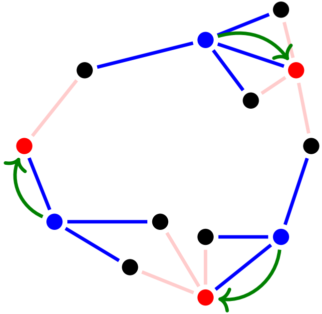

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Because (s, o) is not an improving swap, $\mathrm{cost}(S - s + o) \geq \mathrm{cost}(S)$.

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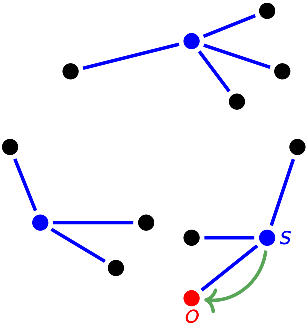

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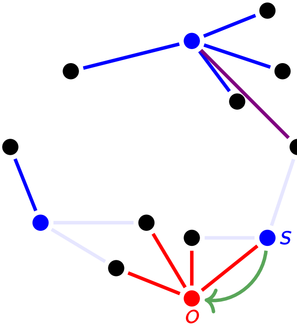

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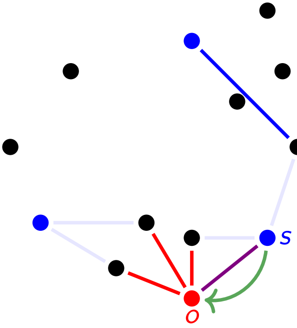

$\mathrm{cost}(S - s + o) \;\leq\; \mathrm{cost}(S) \quad + \quad ???$

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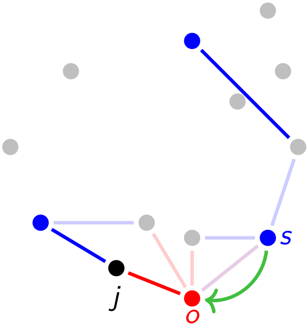

$$\mathrm{cost}(S - s + o) \leq \mathrm{cost}(S) + \sum_{j \in N^*(o)} (o_j - s_j) +$$

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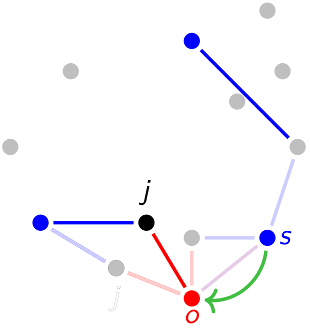

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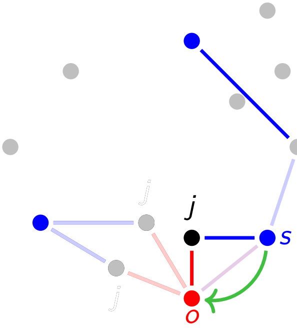

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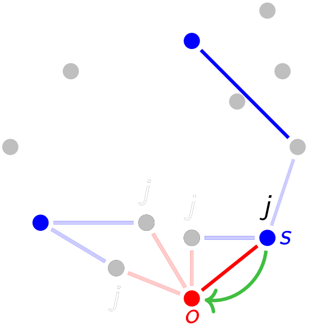

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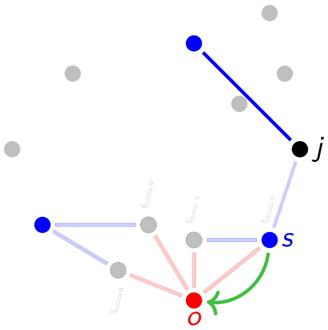

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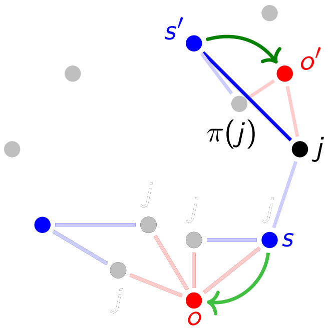

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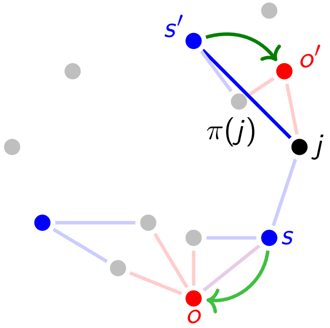

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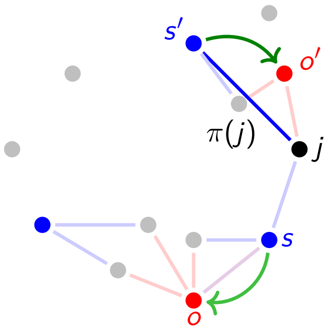

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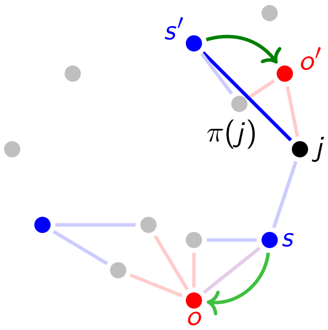

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Permutation π is selected using that $|N^*(o') \cap N(s')| > |N^*(o)|/2$.

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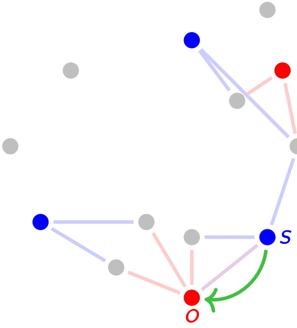

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But $\mathrm{cost}(S - s + o) \geq \mathrm{cost}(S)$, so

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$$(\mathrm{cost}(S^*) - \mathrm{cost}(S)) + (\mathrm{cost}(S^*) + \mathrm{cost}(S^*) + \mathrm{cost}(S) - \mathrm{cost}(S)) \quad \geq \quad 0$$

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Exercise 4:

Argue that each $s \in S$ captures at most two elements from S^* and derive that $\sum_{j \in N^*(o)} (o_j - s_j) + 2 \sum_{j \in N(s)} (o_j + o_{\pi(j)} + s_{\pi(j)} - s_j) \geq 0$.

Short break before Part 2

Exercise 1:

Sort the jobs in decreasing order of the processing time before running Graham's algorithm. Can you prove an approximation ratio better than 2 for this algorithm?

Exercise 2:

How does the inapproximability result for k-center apply to metric instances?

Exercise 3:

Is there an α-approximation with $\alpha < 2$ for the metric k-center?

Exercise 4:

Argue that each $s \in S$ captures at most two elements from S^* and derive that $\sum_{j \in N^*(o)} (o_j - s_j) + 2 \sum_{j \in N(s)} (o_j + o_{\pi(j)} + s_{\pi(j)} - s_j) \geq 0$.

Satisfiability

Boolean formulas

v_i: boolean variable
$\bar{v}_i$: negation of the boolean variable v_i

literal: a variable or its negation

clause: disjunction (OR) of literals, as for instance $v_1 \vee \bar{v}_2 \vee v_3$

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Boolean formula in conjunctive normal form (CNF):

$$\phi = (v_1 \vee \bar{v}_2 \vee v_3)(\bar{v}_1 \vee \bar{v}_3)(v_2 \vee v_3 \vee \bar{v}_4 \vee v_5)(\bar{v}_1 \vee v_4 \vee \bar{v}_5)$$

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assignment for ϕ: function that assigns **True** or **False** to each variable in ϕ

To decide whether there exists an assignment
that satisfies a CNF formula is NP-complete.

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$$\Pr[C \text{ is satisfied}] = 1 - \frac{1}{2^k}$$

Probabilistic $\frac{1}{2}$-approximation for SAT

Algorithm Johnson (ϕ) $\triangleright$ Johnson '74

1. let V be the set of variables in ϕ
2. for each $v \in V$
3. $\quad x_v := \textsc{Rand}(1/2)$
4. return x $\triangleright$ gives a (probabilistic) 0.5-approximation

$\textsc{Rand}(p)$: returns 1 with probability p or 0 with probability $1 - p$.

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Let m be the number of clauses in ϕ.

Then clearly $\mathrm{Exp}[\mathrm{cost}(x)] \geq m/2 \geq \mathrm{OPT}(\phi)/2$.

One more exercise

Proposed algorithm

1. let V be the set of variables in ϕ
2. $s := \textsc{Rand}(1/2)$ ▷ unique coin flip
3. for each $v \in V$
4. $\quad x_v := s$
5. return x

$\textsc{Rand}(p)$: returns 1 with probability p and 0 with probability $1 - p$.

Exercise 5:

Prove that the proposed algorithm is an α-approximation for some α, or argue that the algorithm is not an approximation algorithm.

Integer programming formulations

For a CNF formula ϕ,
V is its set of variables.

For a clause C of ϕ,
$C_0 := \{v_i : \bar{v}_i \in C\}$ and $C_1 := \{v_i : v_i \in C\}$.

C_0 are the negative variables in C and C_1 are the positive variables in C.

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Consider the following integer linear program (IP) built from ϕ:

maximize $\sum_{C \in \phi} z_C$
subject to

$$\begin{array}{rcll} \sum_{v \in C_0}(1 - x_v) + \sum_{v \in C_1} x_v & \geq & z_C & \text{for every } C \in \phi \\ z_C & \in & \{0,1\} & \text{for every } C \in \phi \\ x_v & \in & \{0,1\} & \text{for every } v \in V \end{array}$$

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Solving this IP is equivalent to finding $\mathrm{OPT}(\phi)$.

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The linear relaxation of the IP built from ϕ is:

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Idea

Use the value of $x_v \in [0, 1]$ to decide how to set v to **True** or **False**.

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Algorithm GW (ϕ) ▷ Goemans and Williamson '94

1. solve the LP above obtaining $\hat{z}$ and $\hat{x}$
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Indeed, $\dot{x}$ satisfies at least $0.63 \sum_{C \in \phi} \hat{z}_C \geq 0.63\,\mathrm{OPT}(\phi)$ clauses.

Analysis

For each clause C, let t be the number of literals in C.

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Hence

$$\mathrm{Exp}[Z_C] \;\geq\; 1 - \Big(\frac{t - \hat{z}_C}{t}\Big)^t.$$

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For each clause C with t literals,
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$$\begin{aligned} \mathrm{Exp}[Z_C] &\geq 1 - (\frac{t - \hat{z}_C}{t})^t \\ &= 1 - (1 - \frac{\hat{z}_C}{t})^t \\ &\geq (1 - (1 - \frac{1}{t})^t)\hat{z}_C \end{aligned}$$

because $f(z) = 1 - (1 - \frac{z}{t})^t$ is concave in the interval $[0, 1]$,
and $f(0) = 0$, so $f(z) \geq z\, f(1)$, which implies the last inequality.

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&= 1-(1-\frac{\hat{z}_C}{t})^t \\
&\geq (1-(1-\frac{1}{t})^t)\hat{z}_C \\
&> (1-\frac{1}{e})\hat{z}_C \\
&> 0.63\,\hat{z}_C
\end{aligned}$$

because $(1-\frac{1}{t})^t < \frac{1}{e}$ for every $t \geq 1$.

Euler's number $e = 2.71828$, the base of the natural logarithm.

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The GW algorithm is a 0.63-approximation for MAXSAT.

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If all clauses have k literals,
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So one of the algorithms works better on formulas whose clauses are long,
and the other on formulas whose clauses are short.

Idea

Run both algorithms and output the best solution.

Joining ideas

Algorithm Combined (ϕ) ▷ Goemans and Williamson '94

1. $x_J := \text{JOHNSON}(\phi)$
2. $x_{GW} := \text{GW}(\phi)$
3. let s_J be the number of clauses of ϕ satisfied by x_J
4. let s_{GW} be the number of clauses of ϕ satisfied by x_{GW}
5. if $s_J \geq s_{GW}$ then return x_J
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&\geq \frac{1}{2}\sum_k \sum_{C\in\mathcal{C}_k}((1-\frac{1}{2^k}) + (1-(1-\frac{1}{k})^k)\hat{z}_C) \\
&\geq \frac{1}{2}\sum_k \sum_{C\in\mathcal{C}_k}(1-\frac{1}{2^k} + 1 - (1-\frac{1}{k})^k)\hat{z}_C
\end{aligned}$$

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&\geq \frac{1}{2}\sum_k \sum_{C\in\mathcal{C}_k} \frac{3}{2}\,\hat{z}_C \\
&= \frac{3}{4}\,\mathrm{OPT}(\phi)
\end{aligned}
$$

Conclusions

If you like algorithms and
the use of smart ideas to design beautiful and efficient algorithms,
join the force to study approximation algorithms!

Two books on the subject

Approximation Algorithms, by Vazirani
The Design of Approximation Algorithms, by Williamson and Shmoys

THANK YOU!!!