Second-Order Superintegrable Systems: Geometric Insights via Affine Hypersurfaces

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Abstract: Second-order (maximally) superintegrable systems are a classical area of study in mathematical physics, and beyond. A famous example is the Kepler-Coulomb system, which models both the two-body problem in celestial mechanics and the Hydrogen atom in quantum physics. The classification of second-order superintegrable systems is an open problem, with a complete classification existing only in dimension two (partial results exist in dimension three).

In the first part of the talk, I will introduce the geometric framework, developed jointly with J. Kress and K. Schöbel [1,2,C]. Unlike other techniques, our method is manageable in any dimension. It encodes the systems via "structure tensors". An alternative formulation involves affine connections, offering further insight into the geometric properties of the system [1,3].

The second part of the talk will focus on novel results and applications of the framework. Specifically, I will show how a significant subclass of second-order superintegrable systems (so-called abundant systems) can be realised as affine hypersurfaces, establishing a correspondence between these systems and a certain subclass of affine hypersurface normalisations, which respects conformal rescalings (in preparation with V. Cortés).

Affine hypersurfaces play a key role in Hessian geometry. A Hessian metric is a (Riemannian) metric that can be expressed as the Hessian of a (convex) function with respect to a flat connection. Many non-degenerate second-order superintegrable systems are underpinned by Hessian metrics (with J. Armstrong [B]).

In the final part, I will explore abundant superintegrable systems on flat spaces. Their associated Hessian geometry defines a commutative and associative product structure via the Amari–Chentsov tensor. As a result, these systems can be realised as Manin-Frobenius manifolds, subject to certain simple compatibility conditions [A].

References:

[1] J. Kress, K. Schöbel and A. Vollmer: *An Algebraic Geometric Foundation for a Classification of Superintegrable Systems in Arbitrary Dimension*, J. Geom. Analys. 33, no. 360, 2023

[2] J. Kress, K. Schöbel and A. Vollmer: *Algebraic Conditions for Conformal Superintegrability in Arbitrary Dimension*, Commun. Math. Phys. 405, no. 92, 2024

[3] A. Vollmer: *Torsion-free connections of second-order maximally superintegrable*, Bull. London Math. Soc. 57(2), 2025

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[A] A. Vollmer: *Manifolds with a commutative and associative product structure that encodes superintegrable Hamiltonian systems*, 10 Nov 2024, arXiv: 2411.06418

[B] J. Armstrong and A. Vollmer: *Abundant Superintegrable Systems and Hessian Structures*, 8 Oct 2024, arXiv:2410.05009

[C] J. Kress, K. Schöbel and A. Vollmer: *Superintegrable systems on conformal surfaces*, 14 Mar 2024, arXiv:2403.09191